MINIMAL CLOSED NORMAL SUBGROUPS IN COMPACTLY GENERATED LOCALLY COMPACT GROUPS

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ABSTRACT. We present results of P.-E. Caprace and N. Monod on the existence of minimal closed normal subgroups in compactly generated totally disconnected locally compact groups. Under certain conditions, such subgroups exist and at most finitely many of them are non-abelian.

CONTENTS

1. INTRODUCTION

In the paper [3], P.-E. Caprace and N. Monod discuss the structure of locally compact groups focusing on compactly generated groups. They investigate the existence of minimal closed normal subgroups or, equivalently, of maximal Hausdorff quotients. By first treating the totally disconnected case, an existence result is produced using the solution to Hilbert's fifth problem under the assumption that the ambient group does not contain :

(a) infinite discrete normale subgroup, nor
(b) non-trivial closed normal subgroup which is compact-by-{connected soluble}[1].

In this note, we focus on totally disconnected groups and thus let G be a compactly generated t.d.l.c. group. The result holds if we relax (a) and (b) by merely assuming the absence of non-trivial, discrete or compact, closed normal subgroup in G. In order to obtain minimal closed normal subgroups, we invoke Zorn's lemma, but in fact Proposition 1.3 below shows that the intersection of any filtering family of non-trivial closed normal subgroups is non-trivial; even though checking the same condition for chains would suffice. Assuming G not to have non-trivial compact normal subgroups guarantees the group to act faithfully on its Cayley-Abels graphs as a consequence of Proposition 2.2. Those are connected, locally finite, regular graphs on which G acts transitively. Thanks to the faithfulness, a filtering family of closed normal subgroups has non-trivial intersection as soon as all these subgroups are non-discrete, which is our relaxed assumption, see Proposition 2.3.

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[1] *i.e.* fitting in a short exact sequence $1 \to K \to G \to Q \to 1$ with K compact and Q connected and soluble.

Away from inevitable abelian phenomena like in Example 1.4, there cannot be too many minimal closed normal subgroups. Indeed, in the complementary note [4] by the same authors, they prove that only finitely many minimal closed normal subgroups can be non-commutative. However the abelian ones remain controlled by the locally elliptic radical, see Proposition 1.3. Assuming only (a) and (b), the Theorem 1.5 below shows the result to hold for compactly generated locally compact groups that are non-necessarily totally disconnected.

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1.1. **Preliminaries.** Throughout the paper, we adopt the following conventions. Every locally compact group is assumed to be Hausdorff. A subgroup of a topological group G is called *characteristic* if it is invariant under all topological automorphisms of G. Accordingly, a topological group is called *characteristically simple* if it has exactly two characteristic subgroups. We use the script font to denote certain families of subgroups of a given group: $\mathscr{E}, \mathscr{F}, \mathscr{M}, \mathscr{N}$, etc. On the other hand, $\mathcal{N}_G(H)$ denotes the normalizer in G of a subset H, $\mathcal{Z}_G(H)$ the centralizer of H in G, and $\mathcal{Z}(H) = \mathcal{Z}_H(H)$ denotes the center of H. The identity element is denoted by 1 and the trivial group by $\mathbf{1}$. Finally, whenever a *minimal element M of a family $\mathscr{N}$* of subgroups is mentioned, *e.g.* minimal among the collection of closed normal subgroups, it is implicit that M is minimal in the subfamily $\mathscr{N} \setminus \{\mathbf{1}\}$, hence M is non-trivial as well.

We recall the notion of locally elliptic subgroup introduced by V. P. Platonov [6].

Definition 1.1. A subgroup H of a topological group G is called *locally elliptic* if every finitely generated subgroup of H has compact closure in G. Compact subgroups are obvious and important examples.

Proposition 1.2. Any locally compact group G possesses a unique maximal normal locally elliptic subgroup $\mathrm{Rad}_{\mathscr{LE}}(G)$, called the *locally elliptic radical* of G. Moreover, $\mathrm{Rad}_{\mathscr{LE}}(G)$ is a characteristic closed subgroup and $\mathrm{Rad}_{\mathscr{LE}}(G/\mathrm{Rad}_{\mathscr{LE}}(G)) = \mathbf{1}$.   □

1.2. **The statements.** We now state the results mentioned in the introduction.

Proposition 1.3 (Proposition 2.6, [4]). Let G be a compactly generated t.d.l.c. group without non-trivial compact or discrete normal subgroup. Then,
  (i) Every non-trivial closed normal subgroup contains a minimal one.
 (ii) Let $\mathscr{M}$ be the set of minimal closed normal subgroups and $\mathscr{M}_{\mathrm{na}}$ be the subset of non-abelian ones. Then $\mathscr{M}$ might be infinite but $\mathscr{M}_{\mathrm{na}}$ is finite.
(iii) Each abelian $M \in \mathscr{M}$ is locally elliptic, hence contained in $\mathrm{Rad}_{\mathscr{LE}}(G)$. In particular, if $\mathrm{Rad}_{\mathscr{LE}}(G) = \mathbf{1}$, then $\mathscr{M} = \mathscr{M}_{\mathrm{na}}$ is finite.
(iv) For any proper $\mathscr{E} \subset \mathscr{M}$, the subgroup $N_{\mathscr{E}} = \overline{\langle M \mid M \in \mathscr{E} \rangle}$ is properly contained in G.

Example 1.4. Consider the semi-direct product $G = \mathbf{Q}_p^n \rtimes_p \mathbf{Z}$ where the generator 1 of $\mathbf{Z}$ acts on the n-dimensional $\mathbf{Q}_p$-vector space $\mathbf{Q}_p^n$ by multiplication by p. Then G is a t.d.l.c. group generated by the compact subset $\mathbf{Z}_p^n \times \{1\}$ and has no compact or discrete

normal subgroup other than $\mathbf{1}$. Any one dimensional subspace of $\mathbf{Q}_p^n$ is a minimal closed normal subgroup. If $n \geq 2$, there are uncountably many such abelian subgroups.

Using the solution to Hilbert's fifth problem and some more machinery, one obtains the following result for arbitrary compactly generated locally compact groups. The proof is omitted.

Theorem 1.5 (Theorem B, [4])**.** Let G be a compactly generated locally compact group. Then at least one of the following holds.

- (i) G has an infinite discrete normal subgroup.
- (ii) G has a non-trivial closed normal subgroup which is compact-by-{soluble connected}.
- (iii) There exist non-trivial minimal closed normal subgroups, of which only finitely many are non-abelian.

2. Cayley-Abels graphs

Compactly generated t.d.l.c. groups naturally act on certain connected locally finite regular graphs known as *Cayley-Abels graphs*. The starting point of P.-E. Caprace and N. Monod was to observe that whenever one of the latter actions is faithful, any filtering family of non-discrete closed normal subgroups has non-trivial intersection. We shall see that this is always the case modulo a compact subgroup. The method was inspired by the work [2] of M. Burger and S. Mozes.

Let G be a compactly generated t.d.l.c. group and let $\mathcal{B}(G)$ denote set of compact open subgroups of G. The latter forms a neighbourhood base of the identity thanks to van Dantzig's theorem [5].

Definition 2.1 (Abels, [1])**.** Let $U \in \mathcal{B}(G)$ and C be a compact generating set. Assume C to be bi-U-invariant by replacing it by UCU if necessary. By definition the *Cayley-Abels graph* $\mathfrak{g} = (V_\mathfrak{g}, E_\mathfrak{g})$ associated to this data has vertex set $V_\mathfrak{g} = G/U$ and one connects gU to gcU with an edge for each $g \in G$ and $c \in C$.

Proposition 2.2. The Cayley-Abels graph $\mathfrak{g}$ is connected, locally finite and regular. Moreover, G acts transitively and continuously on $V_\mathfrak{g}$ by left multiplication on the cosets. The kernel of the action is $Q = \cap_{g \in G} gUg^{-1}$, hence G/Q acts faithfully on $\mathfrak{g}$.

Proof. The connectedness of $\mathfrak{g}$ is clear since C generates G. Transitivity is obvious. Since C is compact and U is open, the set $C/U = \{cU \mid c \in C\}$ is finite, thus any vertex has $|C/U|$ neighbours. Here $\mathfrak{g}$ has the discrete topology. Nevertheless, the action is continuous if and only if each vertex stabilizer is open. But clearly the stabiliser of $v = gU$ is $G_v = gUg^{-1}$, hence compact and open, implying continuity. The last statement is now clear. $\square$

Proposition 2.3. Let U and C be as above and suppose G acts faithfully on $\mathfrak{g}$. Then any filtering family of non-discrete closed normal subgroups of G has non-trivial intersection.

Proof. Let $\mathcal{N}$ be a filtering family of non-discrete closed normal subgroups of G. For every vertex $v_0 \in V_\mathfrak{g}$ and every $N \in \mathcal{N}$, the stabiliser of v_0 in N, namely $N_{v_0} = N \cap G_{v_0}$, acts on the set of neighbours of v_0. Let $v_0^\perp$ denote this set and write $F_N < \mathrm{Sym}(v_0^\perp)$ for the permutation subgroup defined by N in this way. We claim that the finite group F_N is independent of the choice of the vertex v_0. For every $v_0, v_1 \in V_\mathfrak{g}$, the subgroups N_{v_1} and N_{v_0} are conjugate. Indeed, transitivity implies that $v_1 = gv_0$ for some $g \in G$, thus

$$N_{v_1} = N \cap G_{gv_0} = gNg^{-1} \cap gG_{v_0}g^{-1} = gN_{v_0}g^{-1},$$

where we used that N is a normal subgroup. Thus, N_{v_0} and N_{v_1} yield isomorphic groups of permutations as claimed.

We now show that F_N is non-trivial. For if N_{v_0} acts trivially on $v_0^\perp$, then $N_{v_0} \subset N_{v_1}$ for all $v_1 \in v_0^\perp$. Recall that N_{v_1} permutes his own neighbours via F_N; this means that N_{v_0} fixes every vertex at distance 2 of v_0. Inductively, N_{v_0} fixes all vertices in $\mathfrak{g}$. By faithfulness, $\mathbf{1} = N_{v_0} = N \cap G_{v_0}$ which proves N to be discrete, a contradiction.

It is clear that $\{F_N \mid N \in \mathcal{N}\}$ is a filtering family of subgroups sitting inside the finite group $\mathrm{Sym}(v_0^\perp)$. In other words, it is a filtering family of finitely many non-trivial subgroups. Therefore, there is a unique minimal non-trivial subgroup F_{N_0} for some $N_0 \in \mathcal{N}$ and

$$\bigcap_{N \in \mathcal{N}} F_N = F_{N_0} \neq \mathbf{1}.$$

Fix $g \neq 1$ in F_{N_0} and let $N_g \subset N_{v_0}$ be the subset of elements acting as g on $v_0^\perp$. The family $\{N_g \mid N \in \mathcal{N}\}$ consists of closed non-empty subsets of the compact subgroup G_{v_0}. This family is filtering; by compactness, it must have non-empty intersection,

$$\emptyset \neq \bigcap_{N \in \mathcal{N}} N_g \subset \bigcap_{N \in \mathcal{N}} N \setminus \mathbf{1}.$$

$\square$

The two previous propositions together show that for compactly generated t.d.l.c. groups the conclusion of Proposition 2.3 holds up to modding out a compact kernel. The next result shows that the latter can be chosen as small as one wishes.

Proposition 2.4 (Proposition 2.5, [3])**.** Let G be a compactly generated t.d.l.c. group and V an identity neighbourhood. Then there is a compact normal subgroup $Q_V \subset V$ such that any filtering family of non-discrete closed normal subgroups of G/Q_V has non-trivial intersection.

Proof. Given a neighbourhood V of the identity, there is $U \in \mathcal{B}(G)$ such that $U \subset V$ and let Q_V be the intersection of all conjugates of U. Then Q_V is compact normal and the group G/Q_V acts faithfully on the Cayley-Abels graph $\mathfrak{g}$ defined with respect to U thanks to Proposition 2.2. One uses Proposition 2.3 to conclude. $\square$

Remarks 2.5. (i) Readily, if G has no compact normal subgroup other than $\mathbf{1}$, then $Q_V = \mathbf{1}$, hence the result holds for G itself.
(ii) If, in addition, the only non-trivial closed normal subgroups of G are the non-discrete ones, then the conclusion of Proposition 2.4 holds for any filtering family of closed normal subgroups. In this situation we can finally apply Zorn's lemma to obtain that any non-trivial closed normal subgroup must contain a minimal one.

3. Minimal closed normal subgroups

In this section we would like to prove the existence of finitely many non-abelian minimal closed normal subgroups as stated in Proposition 1.3. We will start by proving a corollary to the last proposition, which will play an important role later on.

Corollary 3.1 (Corollary to Proposition 2.5, [4])**.** Let G be a compactly generated totally disconnected locally compact group and let $\mathcal{N}$ be a filtering family of closed normal subgroups of G. Then there exist $N \in \mathcal{N}$ and a closed subgroup $Q \trianglelefteq G$ such that $B = \bigcap \mathcal{N}$ is contained in Q, Q/B is compact and $N/N \cap Q$ is discrete.

Proof. Apply Proposition 2.4 to $\overline{G} := G/B$. We get a compact subgroup $\overline{Q} \trianglelefteq \overline{G}$ which lifts to a closed subgroup $Q \trianglelefteq G$ containing B as a cocompact subgroup. Furthermore, the isomorphism $\overline{G}/\overline{Q} \cong G/Q$ implies that any filtering family of closed normal subgroups of G/Q has non-trivial intersection, see (ii) of Remarks 2.5. Recall that a continuous map of locally compact Hausdorff spaces is proper if and only if it is closed and the inverse image of a point is compact. Hence any continuous quotient homomorphism of locally compact groups is closed as soon as its kernel is compact. Therefore, the quotient map $\pi : G/B \to G/Q$ is closed and $\{NQ/Q\}_{N \in \mathscr{N}}$ is a filtering family of closed normal subgroups of G/Q. By the definition of B, $\bigcap \{N/B\}_{N \in \mathscr{N}}$ is trivial in G/B, but this implies that $\bigcap \{NQ/Q\}_{N \in \mathscr{N}}$ is trivial in G/Q (as the map $\bigcap \{NQ/Q\}_{N \in \mathscr{N}} \to \bigcap \{N/B\}_{N \in \mathscr{N}}$, defined by $nQ \mapsto nB$ is injective). This, by Proposition 2.4, implies in turn that for at least one $N \in \mathscr{N}$, $NQ/Q \cong N/N \cap Q$ is discrete, as required. $\square$

We will also use the following lemma.

Lemma 3.2 (Lemma, [4]). *A profinite group which is characteristically simple is isomorphic to the direct product of copies of a given finite simple group.*

We now complete the proof of Proposition 1.3.

Proof of Proposition 1.3. As part (ii) is the most complicated, we will prove it last:

(i) It's an immediate consequence of Proposition 2.4 and Zorn's lemma, as mentioned in the Remarks 2.5.

(iii) Let $M \in \mathscr{M}$ be abelian. Since M is non-discrete, it must contain a non-trivial compact open subgroup. Hence $\mathrm{Rad}_{\mathscr{LE}}(M) \neq \mathbf{1}$. As $\mathrm{Rad}_{\mathscr{LE}}(M)$ is a characteristic subgroup of M, we have $\mathrm{Rad}_{\mathscr{LE}}(M) = M$. In particular, if $\mathrm{Rad}_{\mathscr{LE}}(G) = \mathbf{1}$ then $\mathscr{M} = \mathscr{M}_{\mathrm{na}}$.

(iv) Let $\mathscr{E} \subsetneq \mathscr{M}$ be a proper subset. Denote $M_{\mathscr{E}} = \langle M \mid M \in \mathscr{E} \rangle$ and let $N_{\mathscr{E}} = \overline{M_{\mathscr{E}}}$ be its closure. Observe that for any $M' \in \mathscr{M} \setminus \mathscr{E}$ and $M \in \mathscr{M}$, we have $[M', M] \subseteq M' \cap M$. By the minimality assumption on both M', M we have $M \cap M' = \mathbf{1}$. Hence $[M', M] = \mathbf{1}$ and so $[M', M_{\mathscr{E}}] = \mathbf{1}$, which implies that $[M', N_{\mathscr{E}}] = \mathbf{1}$ as well. Assume for contradiction that $N_{\mathscr{E}} = G$. Then $M' \leq \mathcal{Z}(G)$ and hence M' is abelian. As any subgroup of M' will be normal in G, this implies that M' has to be finite of prime order, but this contradicts our assumption on G.

(ii) As we have seen in (iv), any distinct $M, M' \in \mathscr{M}$ commute. Hence, if $\mathscr{E}_1 \cap \mathscr{E}_2 = \emptyset$ then $N_{\mathscr{E}_1}, N_{\mathscr{E}_2}$ commute.

Consider the family $\mathscr{N} = \{N_{\mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}} \mid \mathscr{F} \subseteq \mathscr{M}_{\mathrm{na}}, |\mathscr{F}| < \infty\}$. It is a filtering family of closed normal subgroups of G, which are non-discrete. Denote $B = \bigcap \mathscr{N}$. By Proposition 2.3, if B is trivial then one of the elements of $\mathscr{N}$ is trivial, *i.e.* $\mathscr{M}_{\mathrm{na}}$ is finite.

Assume from now on that B is non-trivial. By Corollary 3.1, we can find a closed subgroup $Q \trianglelefteq G$ containing B and $N = N_{\mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}} \in \mathscr{N}$ such that Q/B is compact and $N/N \cap Q$ is discrete.

Claim 3.3. Under our notations and assumptions, there exists a $M \in \mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}$ which admits a discrete quotient with a non-trivial center.

Proof. Firstly, we would like to show that $N \leq Q$. Indeed, if $M \in \mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}$ and $M \not\leq Q$ then $M \cap Q = \mathbf{1}$ (by the minimality of M). Hence the map $N \to N/N \cap Q$ restricted to M is injective, with discrete image, so M is discrete, which is a contradiction. Hence $N \leq Q$.

Using the remark above, we see that B commutes with any $M \in \mathscr{M}_{\mathrm{na}}$, and hence $B \leq \mathcal{Z}(N) \leq N \leq Q$. As Q/B is compact and N is closed in Q, so N/B is compact, hence so is $N/\mathcal{Z}(N)$.

Take any compact open subgroup $U \leq N$. As $U \leq \mathcal{N}_N(U)$, $\mathcal{N}_N(U)$ is open. And since $\mathcal{Z}(N) \leq \mathcal{N}_N(U)$, it is also cocompact in N. This implies that $N/\mathcal{N}_N(U)$ is compact and discrete, hence finite, and in particular, U has only finite many conjugates in N. Hence $V = \bigcap_{n \in N} nUn^{-1}$ is a compact open normal subgroup of N. As N/V is discrete, we have $N/V = \langle MV/V \mid M \in \mathscr{M}_{\mathrm{na}} \setminus \mathscr{F} \rangle$. If $\mathcal{Z}(N/V)$ is trivial then $B \leq V$ and hence B is compact. This contradicts the assumption on G. Thus $\mathcal{Z}(N/V)$ is non-trivial and we can find $M_0, \ldots, M_n \in \mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}$ and $m_i \in M_i$ such that $m_0 \cdot \ldots \cdot m_n$ is projected to a non-trivial central element in N/V. Assume that m_0 has non-trivial image. Then, as all elements of $\mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}$ commute pairwise, m_0 has to project to a non-trivial central element of M_0V/V, *i.e.* M_0V/V is discrete with non-trivial center. $\square$

Together with the next claim, we obtain the desired contradiction.

Claim 3.4. Under our notations and assumptions, for all $M \in \mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}$, any discrete quotient of M is centerfree.

Proof. Fix any $M \in \mathscr{M}_{\mathrm{na}} \setminus \mathscr{F}$. As Q/B is compact, so is N/B and hence so is $\overline{BM}/B$. Since $B \leq \mathcal{Z}_G(M)$, we have $\overline{M\mathcal{Z}_G(M)}/\mathcal{Z}_G(M)$ is compact.
Consider now $\pi : G \to G/\mathcal{Z}_G(M)$. We should have $\pi|_M$ injective, as if $\ker(\pi|_M)$ is non-trivial, then it has to be equal to M (by minimality) but this implies that $M \leq \mathcal{Z}_G(M)$, *i.e.* M is abelian, contradicting the choice of M. Now we would like to show that $\overline{\pi(M)}$ is a minimal closed normal subgroup of $G/\mathcal{Z}_G(M)$.

Take any $\overline{P} \leq G/\mathcal{Z}_G(M)$ non-trivial normal closed subgroup. It lifts to a closed subgroup $P \trianglelefteq G$ such that $\mathcal{Z}_G(M) \subsetneq P$. Thus $\mathbf{1} \neq [P, M] \leq P \cap M$, and, by minimality, $M \leq P$. Hence $\overline{\pi(M)} \leq \overline{P}$.

As $\overline{\pi(M)}$ is a minimal closed normal subgroup, it has to be characteristically simple. By Lemma 3.2, $\overline{\pi(M)}$ is isomorphic to a product of a finite simple group S, which is not abelian (as M is not abelian and $\pi|_M$ is injective). Let S denote such a factor. As $\overline{\pi(M)}$ is not abelian, we have $\mathbf{1} \neq [S, \pi(M)] \leq S \cap \pi(M)$. As S is simple, we must have $[S, \pi(M)] = S$. Hence $\pi(M)$ contains the direct sum of all the simple factors of $\overline{\pi(M)}$, call it D. Now, $D' = \pi^{-1}(D) \cap M$, is a normal subgroup of G and contained in M, isomorphic to D and dense in M, by minimality. D' is a direct sum of copies of a simple group, hence any discrete quotient of D' is centerfree. Since D' is dense, it implies that any discrete quotient of M is centerfree. $\square$

References

[1] H. Abels, *Specker-Kompaktifizierungen von lokal kompakten topologischen Gruppen*, Math. Z. **135**, 325-361 (1973/74).

[2] M. Burger, S. Mozes, *Groups acting on trees : from local to global structure*, Inst. Hautes Etudes Sci. Publ. Math. **92** (2000), 113-150.

[3] P.-E. Caprace, N. Monod, *Decomposition locally compact groups into simple pieces*, Math Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97-128.

[4] P.-E. Caprace, N. Monod, *Correction to : "Decomposition locally compact groups into simple pieces"*, in preparation.

[5] D. van Dantzig, *Studien over topologische algebra* (proefschrift). Ph.D. thesis (Groningen, 1931).

[6] V. P. Platonov, *Locally projectively nilpotent radical in topological groups*, (russian), Dokl. Akad. Nauk BSSR **9** (1965), 573-577.

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